How Cinderella and Stepmother Win the Bucket Challenge with Six Buckets

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Introduction

While working on my graduate degree, I was introduced to a problem in an article titled "How Cinderella Won the Bucket Game (and Lived Happily Ever After)". I have always been interested in game theory and my daughter was very interested in Disney princesses at the time, so it seemed like a perfect fit. The article presented a challenge between Cinderella and her evil Stepmother; there are five identical buckets of capacity b placed at the vertices of a regular pentagon.

At the beginning of each round, Stepmother gets one gallon of water from a nearby river and distributes the water arbitrarily among the five buckets. Cinderella then gets to choose any two adjacent buckets, takes them to the nearby river to empty, and then places the buckets back in their original position. Cinderella's goal is to keep the buckets from overflowing and Stepmother's goal is to cause a bucket to overflow. The article posed two questions: For which values of bucket sizes b can Stepmother cause an overflow and win? For which values of bucket size b can Cinderella prevent an overflow continuing the game forever?

Given the five bucket challenge, Stepmother can win for bucket size b when b is less than two gallons. Cinderella can keep the game infinitely running for b values greater than or equal to two gallons. My question while reading about this problem was *why* specifically five buckets? Would the process for Cinderella and Stepmother winning change if we change the number of buckets that are placed around a circle? Would the value of the bucket size b be different for each scenario when the number of buckets is changed?

I decided to look at the scenario of having six buckets evenly spaced around a circle while keeping the questions the same. For which values of bucket sizes b can Stepmother cause an overflow and win? For which values of bucket size *b* can Cinderella prevent an overflow continuing the game forever?

Strategy for Stepmother

There are six empty buckets, $B_1, B_2, B_3, B_4, B_5,$ and B_6, evenly spaced apart around a circle. Stepmother has a one gallon bucket and takes it to a nearby river and fills it with water. Stepmother can place the gallon of water arbitrarily in any of the six buckets. Cinderella can empty any two buckets, but they must be adjacent buckets. Stepmother wins the challenge by causing the water in any bucket to overflow. Given this scenario, for which values of bucket sizes *b* can Stepmother cause an overflow and win?

If the value of *b* is strictly less than one gallon, it's easy to see that Stepmother can win in the first round by simply pouring the whole gallon of water from the river into just one bucket causing an overflow. What gets a little more complicated is when we look beyond *b*<1 to figure out exactly what value of *b* will Stepmother win. Since Cinderella can only dump water from two adjacent buckets, Stepmother distributes the gallon of water into nonadjacent buckets so that the buckets have the same amount of water in each bucket. On the first round, Stepmother distributes the water evenly between $B_1, B_3,$ and B_5, where B_i indicates the position of the bucket in the circle.

This means that $B_1, B_3,$ and B_5 each have $\frac{1}{3}$ gallon of water, while the remaining buckets are empty. Cinderella can now dump any two adjacent buckets of water. She empties B_5 and B_6, then Stepmother has one gallon and places the water so that $B_1, B_3,$ and B_5 have equal amounts of water after pouring. Before Stepmother pours the water, B_1 and B_3 each have $\frac{1}{3}$ gallon of water and B_5 is empty. Stepmother ends up with $\frac{1}{3}\left(1+\frac{2}{3}\right)=\frac{1}{3}+\frac{2}{9}=\frac{5}{9}$ gallon of water in each

$B_1, B_3,$ and B_5 by pouring an additional $\frac{2}{9}$ gallon of water into $B_1$and B_3, while pouring $\frac{5}{9}$ gallon of water into B_5.

Cinderella empties the same buckets as done previously. Stepmother repeats her process and ends up with $\frac{1}{3}\left(1+\frac{2}{3}+\frac{4}{9}\right)=\frac{1}{3}+\frac{2}{9}+\frac{4}{27}=\frac{19}{27}$ gallon of water in each $B_1, B_3,$ and B_5 by pouring an additional $\frac{4}{27}$ gallon of water into $B_1$and B_3, while pouring $\frac{19}{27}$ gallon of water into B_5. This process will continue n times until Stepmother is placing $\frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots+\left(\frac{2}{3}\right)^{n-1}\right)=$ $\frac{1}{3}\left[\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}\right]=1-\left(\frac{2}{3}\right)^n$ which approaches 1 as $n\to\infty$, and so for some small real number such that so $\varepsilon>0$, $B_1, B_3,$ and B_5 each have $1-\varepsilon$ gallon of water.

At this point, Cinderella empties one of the filled buckets and the empty neighboring bucket, say B_5 and B_6. Now, Stepmother splits the gallon of water evenly between B_1 and B_3. This means she adds half a gallon to each bucket that has $1-\varepsilon$ gallons of water currently, hence ending with $\frac{3}{2}-\varepsilon$ in both buckets B_1 and B_3. Cinderella can empty B_1 or B_3 and any other empty neighboring bucket, but this does not matter because Stepmother will win on the next round. Stepmother pours the entire gallon of water into whichever bucket Cinderella did not empty, leaving $\frac{5}{2}-\varepsilon$ and causing an overflow when $b<\frac{5}{2}-\varepsilon$. Since we can make ε arbitrarily small, Stepmother wins when $b<\frac{5}{2}$.

Strategy for Cinderella

Let x_i be the amount of water in B_i after Cinderella has emptied two buckets during any given round. Stepmother pours a gallon of water arbitrarily into the buckets during her round. Let y_i be the amount of water in B_i after Stepmother has arbitrarily poured the gallon of water into the buckets. Let the invariants be $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 2$ and all $x_i < \frac{3}{2}$. These hold true from the first round since Stepmother may only pour one gallon of water and her strategy to win starts with the buckets at $\frac{1}{3}$ gallon of water during her first turn.

We now have $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 < 3$ and all $y_i < \frac{5}{2}$ since $x_i + 1 < \frac{3}{2} + 1 = \frac{5}{2}$. This implies that at most one y_i bucket can be greater than or equal to $\frac{3}{2}$ since the sum of all y_i buckets are strictly less than 3.

If we indeed have one $y_i \geq \frac{3}{2}$, then Cinderella empties that bucket and one of the two neighboring buckets. After this move, all $y_i < \frac{3}{2}$ and $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 < \frac{3}{2} < 2$. Cinderella can continue this process, preserving the given invariants, and thus winning the challenge.

However, what if all $y_i < \frac{3}{2}$? Then depending on which two buckets Cinderella dumps we have the following:

$$y_1 + y_2 + y_3 + y_4 \geq 2 \qquad y_4 + y_5 + y_6 + y_1 \geq 2 \qquad y_2 + y_3 + y_4 + y_5 \geq 2$$

$$y_3 + y_4 + y_5 + y_6 \geq 2 \qquad y_6 + y_1 + y_2 + y_3 \geq 2 \qquad y_5 + y_6 + y_1 + y_2 \geq 2$$

Taking the sum of these six inequalities, we get the following inequality:

$$12 > 4(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) \geq 12$$

Since $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 < 3$, we now have $12 > 4(3) \geq 12$ giving us $12 > 12$ which is a contradiction. This implies that one of the above sums must be less than two. Cinderella dumps whichever two neighboring buckets that keeps $y_1 + y_2 + y_3 + y_4 + y_5 + y_6 < 2$ and all $y_i < \frac{3}{2}$. Stepmother will only have bucket sizes $b < \frac{3}{2}$, and so will never overflow.

Thus, Cinderella wins by preserving the given invariants, causing the challenge to run infinitely when $b \geq \frac{5}{2}$.

n Bucket Strategy for Cinderella

We know that Stepmother wins the six bucket challenge when $b < \frac{5}{2}$, and Cinderella wins when $b \geq \frac{5}{2}$. We also know that Stepmother wins the five bucket challenge when $b < 2$, and Cinderella wins when $b \geq 2$. This shows the answer to the question of whether or not the value of *b* would be different from the value of *b* in the five bucket challenge, is that the value of *b* changed when we changed the number of buckets available.

While working on the six bucket challenge, I found the paper that was referenced in the article I read. While reading the paper "Cinderella Versus the Wicked Stepmother", they mentioned a theorem that states the *b* value of a seven bucket challenge being $\frac{5}{2}$ and that the proof for it could be found in the full version of the paper. Hoping to compare it to the six bucket challenge problem I was working on, my advisor tried finding the full article for insight. Neither one of us could find it online anywhere. After sending emails to authors, we either got no response or that they personally did not have the full paper.

However, the paper did give insight as to what the value of *b* is given *n* buckets provided the rules of the challenge remained the same. Given $n \geq 3$ and Cinderella is able to empty only two adjacent buckets, then the value of *b* can be determined by two cases. If *n* is even, take the sum of the harmonic sequence of $\frac{n-2}{2}$ and 1. If *n* is odd, take the sum of the harmonic sequence of $\frac{n-3}{2}$ and 1. For example, if I want to find the *b* value for a ten bucket challenge, then *n* is even and I would sum the harmonic sequence of $\frac{10-2}{2} = 8$ and 1, so $b = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + 1 = \frac{37}{12}$. If I want the *b* value for a nine bucket challenge, then *n* is odd and I would sum the harmonic sequence of $\frac{9-3}{2} = 3$ and 1, so $b = 1 + \frac{1}{2} + \frac{1}{3} + 1 = \frac{17}{6}$.

Final Remarks

I set out to determine the *b* value at which Stepmother could win the six bucket challenge, which I learned is when $b < \frac{5}{2}$. I also wanted to know at which value of *b* would Cinderella be able to keep the challenge going forever, which ended up being when $b \geq \frac{5}{2}$. In addition, I wanted to know if they would be the same values as in the five bucket challenge. The value of *b* for the five bucket challenge was 2, obviously not the same as $\frac{5}{2}$.

While I was able to determine a winning strategy for Stepmother and Cinderella given six buckets instead of five, surprisingly the strategy for Cinderella winning with six buckets was not as simple as with five buckets. I was also able to determine the value of *b* at which they each win and it is easy to see how the strategy to win could get much more complicated for a larger amount of buckets. Interestingly, I was able to figure out the value of *b* given *n* buckets. The pattern that is created while finding the value of *b* for a given amount of buckets is actually quite intriguing and in a sense, beautiful. Seems almost fitting since the whole challenge is based around Cinderella.

References

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